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Existence and uniqueness of normal shock waves in gas–particle mixtures

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A mixture of a gas and small solid particles is considered which, far upstream, is in a constant equilibrium state, and moves with a constant velocity. The existence of shock waves is investigated in the four possible cases, namely for frozen flow, for two kinds of partly frozen flow, and for equilibrium flow. It is shown that, in all these cases, compressive shocks may exist, if the upstream velocity exceeds the velocity of sound appropriate to the type of flow. Rarefaction shocks are impossible in each case. Moreover, it is shown that the downstream values of the flow parameters are determined uniquely, and the direction of their change is given. Only rather general assumptions concerning the behaviour of the gas are needed. The paper takes into account the influence of the finite particle volume fraction unlike most previous papers on the topic.

1. Introduction

The motion of a gas which contains an appreciable number of small solid particles differs greatly from the motion of the pure gas. This is because a finite time is needed for the velocity and temperature of the particles to adjust to those of the gas. Because of this relaxation, in general, there exists a difference between the velocity and the temperature of the particles and the gas.

In this paper the stationary one-dimensional flow of a mixture is treated. It is presumed that the upstream values of the velocity and the temperature are the same for both constituents.

The structure of shock waves in such a flow was considered by Carrier (1958). According to Carrier the change of the flow conditions is accomplished in two phases. First, there occurs a compressive change in the state of the gas, which is consistent with the usual shock relations for the pure gas. Since this compression takes place in a few mean free paths of the gas molecules, the relaxation processes, which require much larger distances, cause very little change in the state of the particles. Therefore the relaxation processes may be considered as frozen. This frozen shock is followed by the relaxation zone, where the equilibrium is re-established. The extent of this second region depends on the characteristic lengths of the relaxation processes, i.e. on the relaxation lengths. On a scale which is much larger than both relaxation lengths, the frozen shock together with the relaxation zone appears as a so-called equilibrium shock. The change in the state of the mixture, undergoing an equilibrium shock, is given by certain shock relations, too. In order to exhibit the transition from frozen to equilibrium conditions, Carrier calculated and discussed the velocity of the gas as a function of that of the particles, for some characteristic values of the parameters across the whole relaxation zone.

Rudinger (1964) found that some flow variables do not always vary monotonically in the relaxation zone. He also drew attention to the fact that equilibrium shocks may occur at upstream velocities which permit no frozen shock. In such a case the change of the flow conditions is fully dispersed, and only if the scale is chosen suitably does it appear as an equilibrium shock.

Both authors made the assumption that the pure gas follows the perfect gas law. Moreover, they used an approximation, which is mainly to neglect two effects. The first of these is the reduction of the volume of the gas caused by the occupation of a finite volume by the particles. The second effect is the acceleration of the particles produced by the pressure gradient acting on them. Both effects vanish if the particle volume fraction tends to zero.

Rudinger (1965) demonstrated that the approximation discussed above gives rise to an error in the results concerning the frozen shock, as well as in those concerning the equilibrium shock. In the second case, even a small particle volume fraction may cause a considerable error.

At this stage of knowledge it seems necessary to study the effects in a more general way. Therefore in this paper an investigation is made into the possible discontinuous changes of the flow conditions in the mixture, and under what conditions they occur. The number of assumptions is kept at a minimum. Thus only some rather general assumptions concerning the behaviour of the pure gas are used, which are fulfilled by most of the common gases. Moreover, the influence of the finiteness of the particle volume fraction is not neglected.

First, the existence and uniqueness of frozen compressive shock waves are proved. In the following section the corresponding proof is sketched for a partly frozen compressive shock wave, which is characterized by the fact that the temperature of the particles reaches its equilibrium value while the other relaxation process is frozen. In the last section the existence and uniqueness of compressive shock waves, with velocity equilibrium, is investigated. Thus all cases are studied which may occur if, by choosing a suitable scale, the extent of the relaxation zone can be ignored. Moreover, the results are needed as a basis for a general discussion of the qualitative behaviour of the flow variables across the relaxation zone. This will be the subject of another paper.

2. Assumptions and basic equations

Some of the assumptions concerning the properties of the particles which have been used by previous investigators are kept in this paper. These are:

(i) The dimensions of the particles are much larger than the mean free path of the gas molecules.

(ii) All particles consist of the same incompressible material. Their shape is approximately spherical and they have equal size.†

(iii) The particle diameter does not vary, i.e. any mass transfer between the particles and the gas is excluded.

(iv) The temperature gradient in the interior of a particle may be neglected.

(v) Direct interactions between the particles may be neglected.

Since the calculation of the local change of the flow conditions in the neighbourhood of a single particle is an exceedingly complex task, the ensemble of the particles is treated as a second continuum. Then, at each point of the flow field, the velocity of this 'particle continuum' is a mean value of the velocities of all particles being contained in a volume element with dimensions of the order of magnitude of a few inter-particle distances. In the same way, a temperature of the particles may be defined at each point of the flow field. Velocity and state variables of the gas are introduced in an analogous way. Characterization of the state of the mixture is completed by giving the value of the particle volume fraction.

In addition to the assumptions concerning the properties of the particles, it is assumed that the influence of viscosity and heat conduction of the gas is of importance only in the gas–particle interaction processes.

In this paper the one-dimensional stationary flow is considered, its direction being identified with the direction of the positive x-axis. Far upstream ($x \to -\infty$) the mixture is assumed to be in a constant state of equilibrium (index 0). Then the equations of conservation of the mass of the gas, the mass of the particles, the momentum and the energy of the mixture, may be written in the following form (cf. Kraiko & Sternin 1965; Rudinger 1965):

$$\frac{1-\epsilon}{\nu_G} u = \frac{1-\epsilon_0}{\nu_{G0}} u_0, \tag{2.1}$$

$$\epsilon v = \epsilon_0 u_0, \tag{2.2}$$

$$\frac{1-\epsilon}{\nu_G} u^2 + \rho_P \epsilon v^2 + p = \left(\frac{1-\epsilon_0}{\nu_{G0}} + \rho_P \epsilon_0\right) u_0^2 + p_0, \tag{2.3}$$

$$\frac{1-\epsilon}{\nu_G} u \left(h_G + \frac{u^2}{2}\right) + \rho_P \epsilon v \left(c_P T_P + \frac{p}{\rho_P} + \frac{v^2}{2}\right) = \left(\frac{1-\epsilon}{\nu_G} u\right)_0 \left(h_G + \frac{u^2}{2}\right)_0 + \rho_P (\epsilon u)_0 \left(c_P T_P + \frac{p}{\rho_P} + \frac{u^2}{2}\right)_0. \tag{2.4}$$

Herein, ϵ is the particle volume fraction, ν_G the specific volume of the gas, ρ_P the density of the particle material. u and v are the velocities of gas and particles, respectively. The total pressure is denoted by p. Since, on the present assumptions, the random motion of the particles contributes to the pressure only very

† Dropping the assumption of equal size of the particles leads to different relaxation lengths for the different types of particles. The present considerations are valid also for this case if all expressions like 'a length smaller than the relaxation length' are replaced by 'a length smaller than the smallest relaxation length'.

little, the difference between the total pressure of the mixture and that of the gas is negligible (cf. Rudinger 1965). The enthalpy per unit mass of the gas is denoted by h_G, the particle temperature by T_P and the specific heat of the particle material by c_P, which is assumed to be a constant.

The equations of conservation of momentum and energy of the particles across a shock front, where the relaxation processes can be considered as frozen, reduce to (cf. Kraiko & Sternin 1965; Rudinger 1965):

$$\frac{v^2}{2}+\frac{p}{\rho_P} = \frac{u_0^2}{2}+\frac{p_0}{\rho_P}, \tag{2.5}$$

$$T_P = T_{G0}, \tag{2.6}$$

where T_G is the gas temperature.

On the other hand, if the flow is considered in a scale much larger than the relaxation lengths, a discontinuity leads to an equilibrium state, and (2.5) and (2.6) have to be replaced by

$$v \equiv u, \tag{2.7}$$

$$T_P \equiv T_G. \tag{2.8}$$

The behaviour of the gas may be described by an equation of state of the form

$$h_G = h_G(p, s_G), \tag{2.9}$$

where s_G is the entropy per unit mass of the gas. With the aid of Gibbs's relation,

$$T_G ds_G = dh_G - \nu_G dp, \tag{2.10}$$

the quantities h_G and T_G may be expressed as functions of p and ν_G. Thereby the system of equations is closed.

In the following, the volume expansivity of the gas is assumed to be positive:

$$\frac{1}{\nu_G}\left(\frac{\partial \nu_G}{\partial T_G}\right)_p > 0. \tag{2.11}$$

In general, this inequality is satisfied by the common gases. With use of (2.11) and of the conditions of thermal and mechanical stability of the gas some further inequalities may be deduced:

$$\left(\frac{\partial h_G}{\partial \nu_G}\right)_p > 0, \tag{2.12}$$

$$\left(\frac{\partial T_G}{\partial p}\right)_{\nu_G} > 0, \tag{2.13}$$

$$\left(\frac{\partial h_G}{\partial p}\right)_{\nu_G} - \nu_G > 0. \tag{2.14}$$

3. Frozen shock waves

The flow conditions behind a frozen shock wave (index F) are characterized by the equations (2.1)–(2.6). In this case, (2.4) may be simplified with (2.5) and (2.6):

$$h_{GF}+\frac{u_F^2}{2} = h_{G0}+\frac{u_0^2}{2}. \tag{3.1}$$

If the gas does not contain any particles, the equations (2.2), (2.5) and (2.6) are identities. Three equations are left which, with $\epsilon_0 = \epsilon_F = 0$, reduce to the ordinary shock relations relating the quantities u_F, ν_{GF}, p_F to u_0, ν_{G0}, p_0. (See Becker (1968), for example.) Elimination of u_0 and u_F leads to the equation of the Hugoniot curve, i.e. the locus of all states (ν_{GF}, p_F), which can be reached from a fixed state (ν_{G0}, p_0). By comparison of the Hugoniot curve with the isentropes, the following facts may be proved (cf. Serrin 1959):

(i) Only final states with $\nu_{GF} \leqslant \nu_{G0}$ (compression) are characterized by $s_{GF} \geqslant s_{G0}$; i.e. only these states may be reached by a physical system.

(ii) Final states with $\nu_{GF} < \nu_{G0}$ exist only if the upstream velocity u_0 exceeds the upstream velocity a_0 of sound in the gas, i.e. if $u_0 > a_0$.

(iii) The final state (ν_{GF}, p_F) and the final velocity u_F are uniquely determined by fixing the upstream state (ν_{G0}, p_0) and velocity u_0.

In this paper, corresponding statements concerning the flow of the mixture will be proved, using geometrical considerations, as Cowan did for the flow of a pure gas. (See Cowan 1958.)

3.1. *Definition of an Hugoniot curve*

In the present case the six equations (2.1)–(2.3), (2.5), (2.6) and (3.1) contain the ten quantities u_0, ν_{G0}, p_0, ϵ_0 and u_F, ν_{GF}, p_F, ϵ_F, v_F, T_{PF}. Elimination of u_0, u_F, v_F and T_{PF} gives two equations relating ν_{GF}, p_F, ϵ_F to ν_{G0}, p_0, ϵ_0. A discussion corresponding to that of a pure gas would be given by comparison of the projection of this curve into the (ν_{GF}, p_F)-plane with the projections of the isentropes. Since it is not possible to find an explicit relation between ν_{GF} and p_F, such a discussion seems to be very complicated. This difficulty is overcome by eliminating the quantity ν_{GF} instead of ϵ_F. Thus, an 'Hugoniot curve' is obtained in the (ϵ_F, p_F)-plane (in the following the index F is omitted):

$$h_G - h_{G0} = \tfrac{1}{2}(p-p_0)\left(\frac{\nu_G}{1-\epsilon}+\frac{\nu_{G0}}{1-\epsilon_0}\right)L(\epsilon,\epsilon_0), \tag{3.2}$$

where
$$\nu_G = \nu_{G0}\frac{1-\epsilon}{1-\epsilon_0}\left\{1+\frac{\rho_0}{2\epsilon^2}\frac{\epsilon_0^2-\epsilon^2}{1-\epsilon_0}L(\epsilon,\epsilon_0)\right\}, \tag{3.3}$$

$$L(\epsilon,\epsilon_0) = 1-\frac{2\epsilon\epsilon_0}{\epsilon+\epsilon_0}. \tag{3.4}$$

$\rho = \rho_P \nu_G$ is the ratio of the densities of the particle material and the gas.

If one is concerned with a special gas, the function $h_G(\nu_G, p)$ is known. Inserting this function and the equations (3.3)–(3.4) into (3.2) gives the equations of the Hugoniot curve. By definition of an Hugoniot function H as

$$H = 2(h_G-h_{G0})-(p-p_0)\left(\frac{\nu_G}{1-\epsilon}+\frac{\nu_{G0}}{1-\epsilon_0}\right)\mathrm{L}(\epsilon,\epsilon_0) \tag{3.5}$$

this equation is reduced to
$$H = 0. \tag{3.6}$$

3.2. *Isentropes*

Since the temperature of the particles remains constant across a frozen shock, the entropy of the particles, too, does not change. Hence, in this case, the only change of entropy is that of the entropy of the gas, and consequently there is no difference between the isentropes of the mixture and those of the gas. Thus the behaviour of the Hugoniot curve given in the (ϵ, p)-plane has to be compared with that of these isentropes. For this reason some knowledge about the shape of these isentropes in the (ϵ, p)-plane is required. That is, the application of the transformation $\nu_G = \nu_G(\epsilon)$ according to (3.3) to the isentropes given in the form

$$p = p(\nu_G; s_G), \tag{3.7}$$

with s_G as a parameter, has to be studied.

First, the differentiation of (3.7) leads to

$$\left(\frac{\partial p}{\partial \epsilon}\right)_{s_G} = \left(\frac{\partial p}{\partial \nu_G}\right)_{s_G} \frac{d\nu_G}{d\epsilon}. \tag{3.8}$$

The first factor at the right side has negative sign, as may be concluded from its relation to the velocity of sound (denoted by a) in the pure gas:

$$\left(\frac{\partial p}{\partial \nu_G}\right)_{s_G} = -\frac{a^2}{\nu_G^2} < 0. \tag{3.9}$$

Furthermore, it may be shown (see appendix A) that the second factor has negative sign too, i.e.

$$\frac{d\nu_G}{d\epsilon} < 0, \tag{3.10}$$

if the following inequalities hold:

$$\rho_0 \geqslant 4, \tag{3.11}$$

$$0 < \epsilon_0 \leqslant \tfrac{1}{3}. \tag{3.12}$$

The first of these relations is always valid, since the density of a solid material is much greater than that of a gas. With regard to the assumptions explained in the second section, the condition (3.12) is also no severe restriction, since the neglect of direct interactions of the particles requires that the distances between the particles be not too small.

With the help of (3.9) and (3.10) it is concluded from (3.8) that

$$\left(\frac{\partial p}{\partial \epsilon}\right)_{s_G} > 0. \tag{3.13}$$

Further differentiation of (3.8) in view of (3.9) leads to

$$\left(\frac{\partial^2 p}{\partial \epsilon^2}\right)_{s_G} = \frac{a^2}{\nu_G^3}\left(\frac{d\nu_G}{d\epsilon}\right)^2 \{Z_1 - f(\epsilon)\}, \tag{3.14}$$

where

$$Z_1 = \frac{\nu_G^3}{a^2}\left(\frac{\partial^2 p}{\partial \nu_G^2}\right)_{s_G}, \tag{3.15}$$

$$f(\epsilon) = \frac{\nu_G(d^2\nu_G/d\epsilon^2)}{(d\nu_G/d\epsilon)^2}. \tag{3.16}$$

With the assumption that (3.11) holds and that (3.12) may be extended to all values reached by ϵ, i.e.

$$0 < \epsilon \leqslant \tfrac{1}{3}, \tag{3.17}$$

it may be shown (see appendix B) that

$$f(\epsilon) < \tfrac{3}{2}. \tag{3.18}$$

Now the gas is assumed to satisfy the condition

$$Z_1 \geqslant \tfrac{3}{2}.\dagger \tag{3.19}$$

Using (3.18) and (3.19) it is concluded from (3.14) that

$$\left(\frac{\partial^2 p}{\partial \epsilon^2}\right)_{sG} > 0. \tag{3.20}$$

Further, in view of (2.12) and (2.14), (2.10) yields

$$\left(\frac{\partial s_G}{\partial p}\right)_\epsilon = \frac{1}{T_G}\left\{\left(\frac{\partial h_G}{\partial p}\right)_{v_G} - v_G\right\} > 0, \tag{3.21}$$

$$\left(\frac{\partial s_G}{\partial \epsilon}\right)_p = \frac{1}{T_G}\left(\frac{\partial h_G}{\partial v_G}\right)_p \frac{dv_G}{d\epsilon} < 0. \tag{3.22}$$

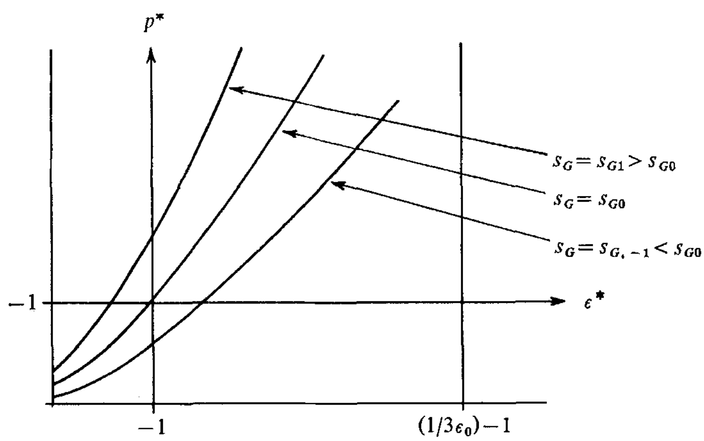

FIGURE 1. Qualitative behaviour of the isentropes (the domains $\epsilon^* < -1$ and $\epsilon^* > (1/3\epsilon_0) - 1$ are excluded by (3.17)).

The qualitative shape of the isentropes following from (3.13), (3.20) and (3.21) is shown in figure 1. Using the transformation

$$\left.\begin{aligned} \epsilon^* &= \frac{\epsilon}{\epsilon_0} - 1, \\ p^* &= \frac{p}{p_0} - 1, \end{aligned}\right\} \tag{3.23}$$

the quantities ϵ^* and p^* are introduced as co-ordinates in this figure, as well as in the following.

† Note that, if the gas follows the perfect gas law, the quantity Z_1 is given by $Z_1 = 1 + \gamma$, where γ is the ratio of the specific heats of the gas. Hence (3.19) is satisfied well by perfect gases.

3.3. *Behaviour of the Hugoniot function on certain curves in the (ϵ^*, p^*)-plane*
On using (2.10), (3.3), (3.4) and (3.23) the differential of the Hugoniot function (3.5) is

$$dH = 2T_G\, ds_G + p_0 p^* M\, dN, \tag{3.24}$$

where the quantity M has the form

$$M = \frac{\nu_{G0}(\epsilon_0 - \epsilon)}{\epsilon^2(1-\epsilon_0)(\epsilon+\epsilon_0)}\left\{2\epsilon^3 + \frac{1}{2}\frac{\rho_0}{1-\epsilon_0}(4\epsilon_0\epsilon^3 + (1-2\epsilon)(\epsilon+\epsilon_0)^2)\right\}.$$

In the following only the sign of M is needed; for $\epsilon < \frac{1}{2}$ this is given by

$$\operatorname{sgn} M = -\operatorname{sgn}\epsilon^*. \tag{3.25}$$

The differential dN introduced in (3.24) is

$$dN = \frac{dp^*}{p^*} + \left(\frac{2}{1+\epsilon^*} - \frac{1}{2+\epsilon^*} - \frac{1}{\epsilon^*}\right) d\epsilon^*. \tag{3.26}$$

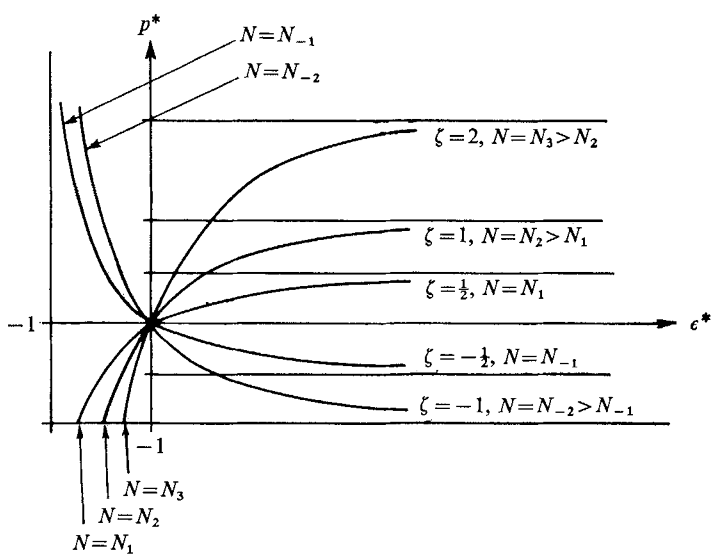

FIGURE 2. Shape of the curves N = const. for some values of ζ.

Hence the quantity N is a constant on the curves

$$p^* = \zeta\frac{\epsilon^*(\epsilon^*+2)}{(\epsilon^*+1)^2}, \tag{3.27}$$

with

$$\zeta = \frac{1}{2}\left\{\left(\frac{\partial p^*}{\partial \epsilon^*}\right)_{N=\text{const}}\right\}_{\epsilon^*=0}. \tag{3.28}$$

The shape of these curves is shown in figure 2. The direction of increasing values of N is inferred from

$$\left(\frac{\partial N}{\partial p^*}\right)_{\epsilon^*} = \frac{1}{p^*}.$$

The following statements concerning the properties of the Hugoniot function may be derived from the foregoing considerations:

(i) *Behaviour of $H(\epsilon^*, p^*)$ in the initial point.* In the point $\epsilon^* = p^* = 0$ the Hugoniot function is equal to zero:

$$H(0,0) = 0. \tag{3.29}$$

(ii) *Behaviour of $H(\epsilon^*, p^*)$ along isochors and isobars.* It may be deduced from (3.24) that

$$\left(\frac{\partial H}{\partial \epsilon^*}\right)_{p^*} = 2T_G\left(\frac{\partial s_G}{\partial \epsilon^*}\right)_{p^*} + p_0 p^* M \left(\frac{\partial N}{\partial \epsilon^*}\right)_{p^*}, \tag{3.30}$$

$$\left(\frac{\partial H}{\partial p^*}\right)_{\epsilon^*} = 2T_G\left(\frac{\partial s_G}{\partial p^*}\right)_{\epsilon^*} + p_0 p^* M \left(\frac{\partial N}{\partial p^*}\right)_{\epsilon^*}. \tag{3.31}$$

On the other hand, it is inferred from (3.25) and (3.26) that

$$M\left(\frac{\partial N}{\partial \epsilon^*}\right)_{p^*} = \frac{-2M}{\epsilon^*(\epsilon^*+1)(\epsilon^*+2)} > 0,$$

$$p^*\left(\frac{\partial N}{\partial p^*}\right)_{\epsilon^*} = 1.$$

Using these results, the relation (3.25), and the inequalities (3.21) and (3.22), the following conclusions concerning the behaviour of $H(\epsilon^*, p^*)$ along isochors and isobars may be drawn from (3.30)–(3.31):

$$\left(\frac{\partial H}{\partial \epsilon^*}\right)_{p^*} < 0, \quad \text{for} \quad p^* \leqslant 0, \tag{3.32}$$

$$\left(\frac{\partial H}{\partial p^*}\right)_{\epsilon^*} > 0, \quad \text{for} \quad \epsilon^* \leqslant 0. \tag{3.33}$$

(iii) *Behaviour of $H(\epsilon^*, p^*)$ along the curves $N = const.$* On the curves $N = \text{const.}$ the relation (3.24) reduces to

$$dH = 2T_G ds_G.$$

That is, the sign of the differential of H agrees with that of s_G.

By considering the isentropes (figure 1) and the curves $N = \text{const.}$ (figure 2) it may be seen that, in the first and third quadrant, there exist points in which a curve of one kind touches one of the other kind. In figure 3 such a point B is marked. Along the part OB of the curve $N = N_3$ the entropy s_G increases with increasing ϵ^*, while in B the entropy s_G begins to decrease. In the second and fourth quadrant, along a curve $N = N_{-2}$, the entropy increases with increasing p^*.

Denoting the locus of touching points B by C_B it may be said: if one travels away from the origin, along a curve $N = \text{const.}$, on the second-quadrant side of C_B one finds *increasing* values of s_G, and hence of H, while on the fourth-quadrant side of C_B one finds *decreasing* values of these functions.

(iv) *Behaviour of $H(\epsilon^*, p^*)$ along isentropes.* On isentropes the relation (3.24) reduces to

$$dH = p_0 p^* M\, dN.$$

Hence, taking account of (3.25), it may be concluded that:

$$\operatorname{sgn} dH = \begin{cases} +\operatorname{sgn} dN, & \text{in the 2nd and 4th quadrant,} \\ -\operatorname{sgn} dN, & \text{in the 1st and 3rd quadrant.} \end{cases} \tag{3.34}$$

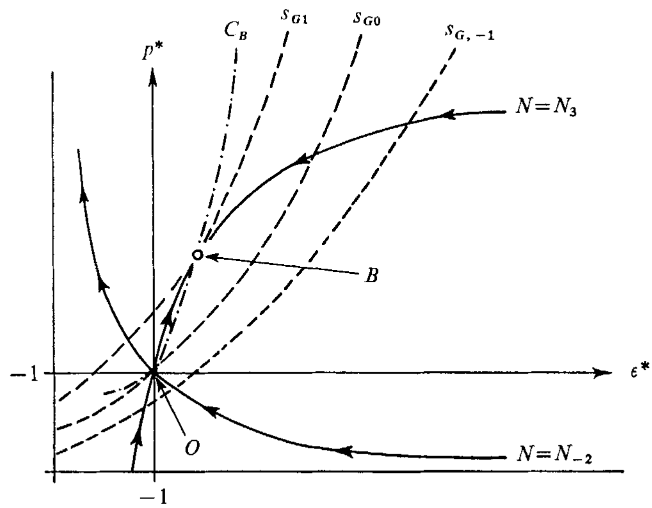

FIGURE 3. Behaviour of the entropy s_G and of the function H along two curves N = const. (s_G and H increase in direction of the arrows).

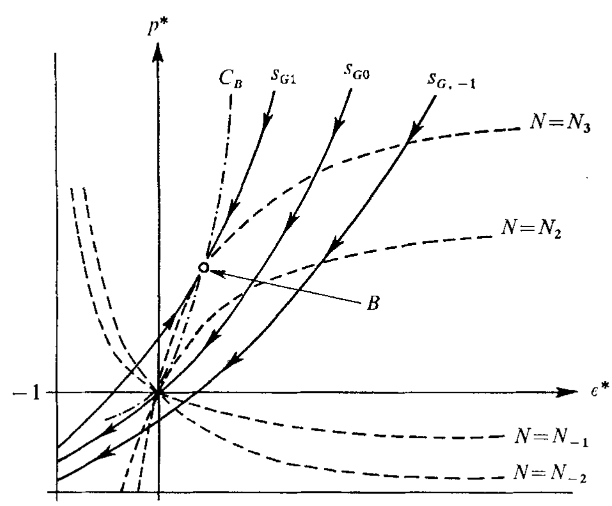

FIGURE 4. Behaviour of the Hugoniot function H along isentropes (H increases in direction of the arrows).

Regarding figure 4, the following statements can be made. In the first and third quadrant, along the isentropes with $s_G \leqslant s_{G0}$, the quantity N increases with increasing ϵ^*. Furthermore, in the first quadrant, along the isentropes with $s_G > s_{G0}$, the value of N decreases with increasing ϵ^* up to the intersection point B of the isentrope and C_B, whereas from the point B the quantity N increases with increasing ϵ^*. When (3.34) is taken into account, these results lead to the behaviour of H, as indicated by the arrows in figure 4. Only the following statements are needed below: in the first quadrant, on travelling along an isentrope away from the origin, one finds *increasing* values of H on the second-quadrant side of C_B, and *decreasing* values of H on the fourth-quadrant side of C_B; in the third quadrant, along $s_G = s_{G0}$, the function H *decreases* with increasing ϵ^*.

3.4. *The shape of the Hugoniot curve*

The results of the last section enable one to make some remarks about the locus at which the Hugoniot function is equal to zero, i.e. the locus of the Hugoniot curve.

To start with the first quadrant, consider the boundary of the shaded area in figure 5 (*a*). This boundary consists of parts of two isentropes, of a curve $N = \text{const.}$,

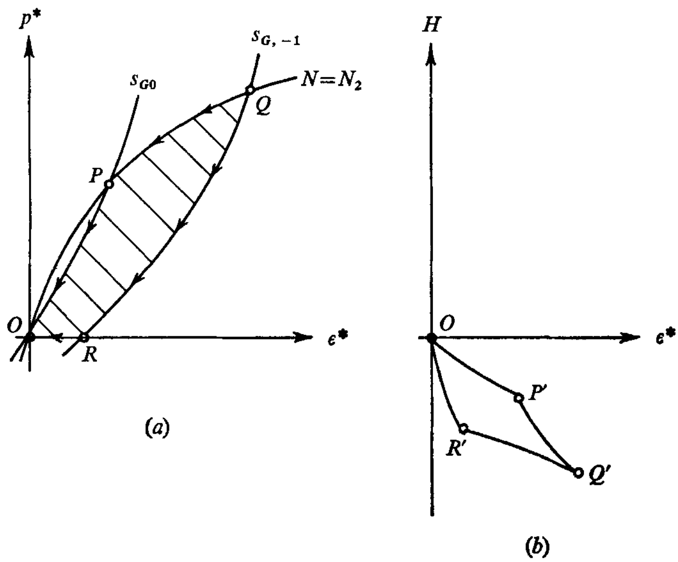

FIGURE 5. (*a*) Behaviour of H along the boundary $OPQRO$ in the (ϵ^*, p^*)-plane. (*b*) The curve $OP'Q'R'O$ in the (ϵ^*, H)-plane.

and of the ϵ^*-axis. The behaviour of the Hugoniot function $H(\epsilon^*, p^*)$ along this curve follows from the considerations given in the last section, and in the figure it is indicated by arrows. The boundary $OPQRO$ may be interpreted as the projection into the (ϵ^*, p^*)-plane of a curve on the surface $H = H(\epsilon^*, p^*)$, given in an (ϵ^*, p^*, H)-space. Hence the projection of the same curve into the (ϵ^*, H)-plane has the shape $OP'Q'R'O$ shown in figure 5 (*b*). Contrary to the figure, the value of H at R' may exceed that at P'. But in both cases H does not vanish along $OP'Q'R'O$, and hence along $OPQRO$, except at the point O.

A similar consideration leads to the correspondence between the boundary $OBPO$ of the shaded area in figure 6 (*a*) and the curve $OB'P'O$ in figure 6 (*b*).

Hence in the first quadrant, on each curve $N = \text{const.}$ cutting the isentrope $s_G = s_{G0}$, there exists one and only one point, other than the origin O, where $H = 0$. This point is situated between the intersection point B with C_B and the intersection point P with the isentrope $s_G = s_{G0}$. On the curves $N = \text{const.}$, which do not cut this isentrope, $H < 0$ in all the first quadrant except at O.

To continue the study with the third quadrant, consider the boundaries of the shaded areas in figure 7 (*a*). They consist of parts of isochors, of isobars and of

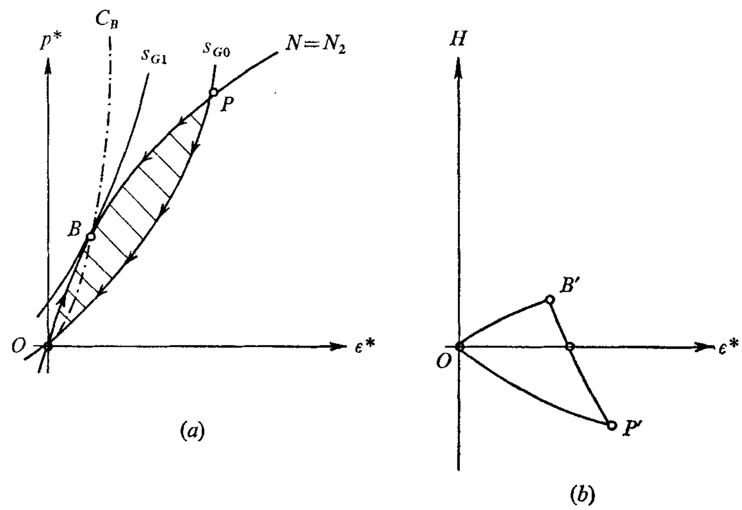

FIGURE 6. (*a*) Behaviour of H along the boundary $OBPO$ in the (ϵ^*, p^*)-plane. (*b*) The curve $OB'P'O$ in the (ϵ^*, H)-plane.

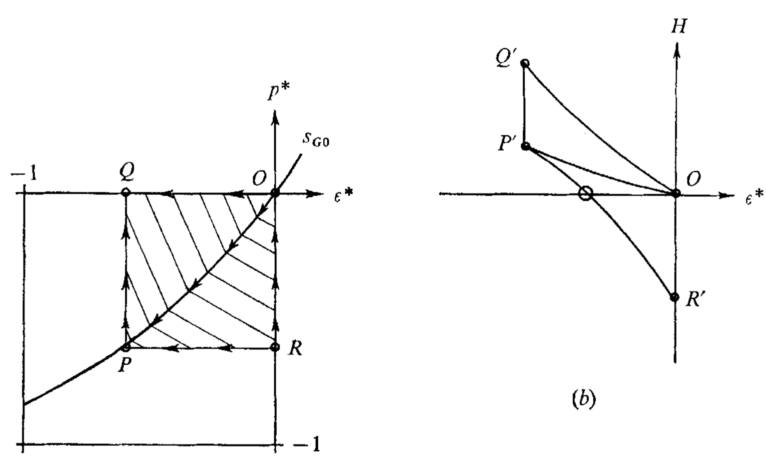

FIGURE 7. (*a*) Behaviour of H along $OPRO$ and along $OPQO$ in the (ϵ^*, p^*)-plane. (*b*) The curves $OP'R'O$ and $OP'Q'O$ in the (ϵ^*, H)-plane.

the isentrope $s_G = s_{G0}$. As figure 7 (*b*) shows, on each isobar $p^* = \text{const.}$, there exists exactly one point where $H = 0$. This point is situated between the intersection point P with the isentrope $s_G = s_{G0}$ and the intersection point R with the p^*-axis.

In the second and fourth quadrant, the function $H(\epsilon^*, p^*)$ varies monotonically along the curves $N = \text{const.}$ Hence H does not vanish except at the origin.

Thus it is shown that the Hugoniot curve $H = 0$ has the qualitative shape given in figure 8.

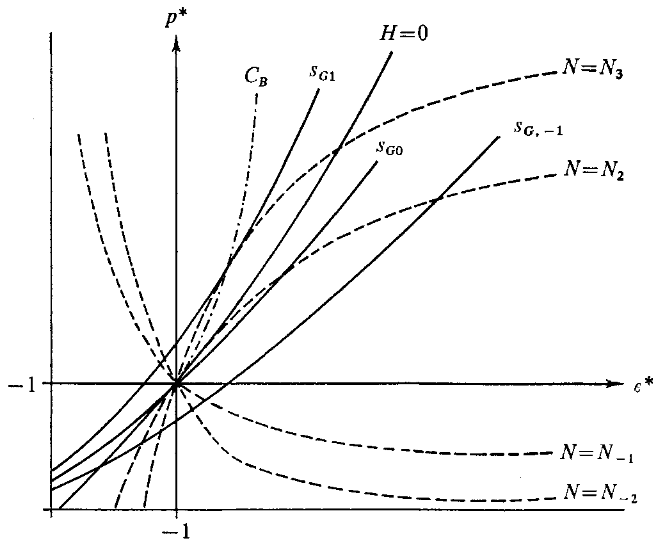

FIGURE 8. Qualitative shape of the Hugoniot curve $H = 0$.

Along the Hugoniot curve in the vicinity of the origin, the deviation of s_G from the value s_{G0} is of third order in ϵ^*:

$$s_G - s_{G0} \approx \frac{a_0^2}{12T_{G0}}(\rho+E)_0^3\,(Z_1-G_1)_0\,\epsilon^{*3}, \tag{3.35}$$

where $E = \epsilon/(1-\epsilon)$ is the ratio of the volume fractions of the particles and the gas, and

$$\left.\begin{aligned} G_1 &= \frac{3E(\rho-1)}{(\rho+E)^2} \\ &= 1-\frac{\rho(\rho-E)+E(\rho+E)}{(\rho+E)^2}. \end{aligned}\right\} \tag{3.36}$$

Since the density ratio ρ exceeds the value 1, while the ratio of the volume fraction is smaller than 1 if $\epsilon < \frac{1}{2}$, it may be concluded from the second form given for G_1:

$$G_1 < 1. \tag{3.37}$$

Hence, according to (3.35), in the vicinity of the origin the qualitative shape of the Hugoniot curve is that shown in figure 8, even if the condition (3.19) does not hold but only the weaker condition

$$Z_{10} \geqslant G_{10}. \tag{3.38}$$

But it must be pointed out that this difference has no physical meaning.

3.5. *Existence and uniqueness of compressive shock waves*

From figure 8 it can be deduced that on the Hugoniot curve

$$s_G \gtrless s_{G0} \quad \text{for} \quad \epsilon^* \gtrless 0. \tag{3.39}$$

Since only final states with $s_G \geqslant s_{G0}$ may be reached physically, it follows from (3.39) that a discontinuous change of state leads to a positive value of ϵ^*; i.e. only compression shocks but no rarefaction shocks are possible (see (3.23) and (A 3)).

For a given upstream state of a mixture, say $(\epsilon_0, p_0, \nu_{G0})$, the Hugoniot curve giving all possible downstream values (ϵ^*, p^*) is uniquely determined. Using the equations (2.1)–(2.3), (2.5) and the definitions (3.23), one finds the following relation between p^* and ϵ^*:

$$p^* = \frac{\rho_P u_0^2}{2p_0} \frac{\epsilon^*(\epsilon^*+2)}{(\epsilon^*+1)^2}. \tag{3.40}$$

Comparison of (3.40) with the equation (3.27) of the curves N = const. shows that by choosing a certain upstream velocity u_0 exactly one of these curves is selected. The downstream values ϵ^* and p^* are represented by the co-ordinates of the intersection point of this special curve and of the Hugoniot curve. Since on each curve N = const., at most one point is situated where $H = 0$ and $s_G > s_{G0}$, in this way the quantities ϵ^* and p^* are determined uniquely.

From figure 8 it may be concluded that exactly one point of this kind exists, if in the origin O the slope of the selected curve N = const. exceeds that of the isentrope $s_G = s_{G0}$, i.e. if

$$\left\{\left(\frac{\partial p^*}{\partial \epsilon^*}\right)_{N=\text{const}}\right\}_0 > \left\{\left(\frac{\partial p^*}{\partial \epsilon^*}\right)_{s_G=s_{G_0}}\right\}_0. \tag{3.41}$$

From this, the following condition for the upstream velocity u_0 is deduced, using (3.27), (3.28), (3.40), (3.8), (3.9), (3.23) and (3.3):

$$u_0^2 > \left(1+\frac{E_0}{\rho_0}\right)a_0^2.$$

The right-hand side of this inequality is the square of the velocity of sound in the mixture, in which the relaxation processes are frozen (see Kraiko & Sternin 1965). Denoting this frozen velocity of sound by b, one may write

$$u_0^2 > b_0^2. \tag{3.42}$$

This condition corresponds exactly to that which is known for the existence of a compressive shock wave in a pure gas. But attention should be paid to the fact that, since the frozen velocity of sound in the mixture exceeds the velocity of sound in the gas, the presence of the particles requires a larger upstream velocity u_0 for a frozen shock to exist. This effect is caused by the acceleration of the particles by the action of the pressure gradient, and it vanishes if the influence of the finite particle volume fraction is neglected.†

The final values of ϵ^* and p^* are positive, and hence (on reintroducing the index F)

$$\epsilon_F > \epsilon_0, \tag{3.43}$$

$$p_F > p_0. \tag{3.44}$$

† This result is not contained in Rudinger's (1965) paper because he argued that, in the case of frozen shocks, the influence of finite ϵ on the flow conditions in the gas is much smaller than in the case of equilibrium shocks. Hence he neglected this influence in the first case.

Taking into account the properties of $\nu_G(\epsilon)$ leads to the conclusion that ν_{GF} is uniquely determined too and that, as mentioned above,

$$\nu_{GF} < \nu_{G0}. \tag{3.45}$$

On the assumption that (3.12) and (3.17) hold, the function L defined by (3.4) is positive, as the following transformation shows:

$$L(\epsilon, \epsilon_0) = \frac{1}{\epsilon+\epsilon_0}\{\epsilon_0(1-\epsilon)+\epsilon(1-\epsilon_0)\}.$$

Hence it may be deduced from (2.1)–(2.3), (2.5) and (3.44) that u_F is uniquely determined too and that

$$u_F < u_0. \tag{3.46}$$

Further, it follows from (2.2) and (3.43) that

$$v_F < u_0. \tag{3.47}$$

The examination of the final state of the mixture is completed by a relation between v_F and u_F, which is derived from (2.1)–(2.2):

$$v_F = \frac{\phi(\epsilon_0)}{\phi(\epsilon_F)}u_F, \tag{3.48}$$

where

$$\phi(\epsilon) = \frac{\epsilon}{1-\epsilon}\nu_G(\epsilon).$$

In view of (3.3) differentiation of this function leads to:

$$\phi'(\epsilon) = \frac{-1}{\rho_0\epsilon^2(1-\epsilon_0)^2}\{(\rho_0(\tfrac{1}{2}-\epsilon_0)-(1-\epsilon_0))\epsilon^2+\tfrac{1}{2}\rho_0\epsilon_0^2\}.$$

Since in the curly brackets the factor of ϵ^2 is positive on the assumptions (3.11) and (3.12) (cf. appendix A), $\phi(\epsilon)$ decreases monotonically with increasing ϵ. Thus (3.43) and (3.48) give

$$v_F > u_F. \tag{3.49}$$

4. Partly frozen shock waves with temperature equilibrium

In this section the length which is characteristic for the equilibration of the temperatures is assumed to be much smaller than the length which is characteristic for the equilibration of the velocities by the action of viscous drag. In this case, in a suitably chosen scale, only the velocities of both media differ, while the temperatures are always the same. Hence also a discontinuity is partly frozen. The flow conditions at the downstream side of such a partly frozen shock (index II) are characterized by (2.1)–(2.5) and (2.8). In this case (2.4) may be simplified with (2.1), (2.2) and (2.8):

$$\left(h_G+\eta_0 c_P T_G+\frac{u^2}{2}\right)_{\text{II}} = \left(h_G+\eta_0 c_P T_G+\frac{u^2}{2}\right)_0. \tag{4.1}$$

The ratio of the mass of the particles to that of the gas is denoted by η:

$$\eta = E\rho = \frac{\epsilon}{1-\epsilon}\rho_P\nu_G.$$

Attention should be paid to the fact that only (2.8) and (4.1) differ from the corresponding equations for frozen shocks. The difference in these two equations, which have the meaning of energy balances, is a consequence of the over-all equilibrium of the temperatures. Because of the similarity of the relations characterizing completely and partly frozen shocks, the proof given in the third section needs only minor modifications in order to fit to the present case. This will be sketched in the following.

An Hugoniot function may be defined by

$$H = 2(h_G - h_{G0}) + 2\eta_0 c_P (T_G - T_{G0}) - (p - p_0)\left\{\frac{v_G}{1-\epsilon} + \frac{v_{G0}}{1-\epsilon_0}\right\} L(\epsilon, \epsilon_0). \quad (4.2)$$

(From now on the index II is dropped.) Contrary to a frozen shock, which causes only a change of the entropy s_G of the gas, a partly frozen shock also gives rise to a change of the entropy s_P of the particles, which is given by

$$T_P \, ds_P = c_P \, dT_P. \quad (4.3)$$

If the upstream and downstream sides of the shock are denoted by + and − respectively, the whole rate of entropy production across the shock is:

$$\left(u\frac{1-\epsilon}{v_G} s_G + v\epsilon\rho_P s_P\right)^+ - \left(u\frac{1-\epsilon}{v_G} s_G + v\epsilon\rho_P s_P\right)^- = u_0\left(\frac{1-\epsilon}{v_G} + \epsilon\rho_P\right)_0 (s^+ - s^-), \quad (4.4)$$

where the quantity s is defined by:

$$ds = \frac{1}{1+\eta_0} ds_G + \frac{\eta_0}{1+\eta_0} ds_P. \quad (4.5)$$

Note that the quantity s does not have the meaning of an entropy of the mixture. Such an entropy cannot be defined, since both components move with different velocities.

Only those partly frozen shocks may exist physically for which the whole rate of entropy production given by (4.4) is positive or zero. From this one may conclude that the quantity s must increase or remain constant. Hence, in this case, the curves $s =$ const. play the same part as the isentropes $s_G =$ const. did in the case of frozen shocks. For this reason, the shape of the curves $s =$ const. must be studied in the (ϵ^*, p^*)-plane.

Using (2.10), (4.3), (2.8) and (A 3) one deduces from (4.5)

$$\left(\frac{\partial p}{\partial \epsilon}\right)_s = \left(\frac{\partial p}{\partial v_G}\right)_s \frac{dv_G}{d\epsilon} = -\frac{a^2}{v_G^2}\frac{1+\eta_0\delta}{1+\gamma\eta_0\delta}\frac{dv_G}{d\epsilon} > 0. \quad (4.6)$$

The quantity δ is given by $\delta = c_P/c_p$, where c_p is the specific heat of the gas at constant pressure. Further differentiation leads to

$$\left(\frac{\partial^2 p}{\partial \epsilon^2}\right)_s = \frac{a^2}{v_G^3}\left(\frac{dv_G}{d\epsilon}\right)^2 \left\{Z_2 - \frac{1+\eta_0\delta}{1+\gamma\eta_0\delta} f(\epsilon)\right\}, \quad (4.7)$$

$$Z_2 = \frac{v_G^3}{a^2}\left(\frac{\partial^2 p}{\partial v_G^2}\right)_s. \quad (4.8)$$

The relation (4.7) corresponds exactly to (3.14). Hence, on the assumptions (3.11), (3.12), (3.17) and

$$\frac{1+\gamma\eta_0\delta}{1+\eta_0\delta}Z_2 \geqslant \tfrac{3}{2},\dagger \tag{4.9}$$

corresponding to (3.19), it may be shown that

$$\left(\frac{\partial^2 p}{\partial \epsilon^2}\right)_s > 0. \tag{4.10}$$

Further, on taking account of (2.11)–(2.14) it may be shown from (4.5), (4.3) and (2.8) that

$$\left(\frac{\partial s}{\partial p}\right)_\epsilon = \frac{1}{(1+\eta_0)\,T_G}\left\{\left(\frac{\partial h_G}{\partial p}\right)_{v_G} - v_G + \eta_0 c_P\left(\frac{\partial T_G}{\partial p}\right)_{v_G}\right\} > 0, \tag{4.11}$$

$$\left(\frac{\partial s}{\partial \epsilon}\right)_p = \frac{1}{(1+\eta_0)\,T_G}\left\{\left(\frac{\partial h_G}{\partial v_G}\right)_p + \eta_0 c_P\left(\frac{\partial T_G}{\partial v_G}\right)_p\right\}\frac{dv_G}{d\epsilon} < 0. \tag{4.12}$$

Because of the validity of (4.6) and (4.10)–(4.12) the qualitative shape of the curves $s = \text{const.}$ is the same as that of the curves $s_G = \text{const.}$ discussed in the third section.

By introduction of the quantity s, the differential of the Hugoniot function defined in (4.2) may be written as:

$$dH = 2(1+\eta_0)\,T_G\,ds + p_0 p^* M\,dN. \tag{4.13}$$

From this point the considerations may be completed in an analogous way to that for frozen shocks. It follows that a partly frozen compressive shock with temperature equilibrium exists and is uniquely determined, if (3.11), (3.12), (3.17) and (4.9) hold.

Along the Hugoniot curve in the vicinity of the origin, the deviation of s from the value s_0 is of the third order in ϵ^* (cf. (3.35)):

$$s - s_0 \approx \frac{a_0^2}{12(1+\eta_0)\,T_{G0}}(\rho+E)_0^2\,(Z_2-G_2)_0\,\epsilon^{*3}, \tag{4.14}$$

where

$$G_2 = \frac{1+\eta\delta}{1+\gamma\eta\delta}G_1. \tag{4.15}$$

Taking account of (3.37) and of $\gamma > 1$ it may be concluded from (4.15) that

$$G_2 < 1. \tag{4.16}$$

Hence, in the case of extremely weak shock waves the assumption (4.9) may be replaced by the weaker condition

$$Z_{20} \geqslant G_{20}, \tag{4.17}$$

† Note that, if the gas follows the perfect gas law, the left-hand side of (4.9) is given by

$$\frac{1+\gamma\eta_0\delta}{1+\eta_0\delta}Z_2 = 1+\gamma\frac{1+\eta_0\delta}{1+\gamma\eta_0\delta}.$$

Hence (4.9) is satisfied well in this case.

which corresponds to (3.38). As in the case of frozen shock waves, the difference between (3.19) and (3.38), so here that between (4.9) and (4.17) has no physical meaning.

For a partly frozen shock with temperature equilibrium the condition for the upstream velocity takes the form

$$u_0^2 > c_{20}^2, \tag{4.18}$$

where

$$c_2^2 = \left(1+\frac{E}{\rho}\right)\frac{1+\eta\delta}{1+\gamma\eta\delta}a^2. \tag{4.19}$$

The quantity c_2 is the velocity of sound in the mixture, in which the velocity equilibration by viscous drag is frozen, while the temperatures of both components are always the same.† Thus also the condition (4.18) corresponds exactly to that known for the existence of a compressive shock in a pure gas. Since $\gamma > 1$, the quantity c_2 is smaller than the frozen velocity of sound, and hence a partly frozen shock is possible at a lower upstream velocity u_0 than a frozen shock. That is because, in the scale chosen, a continuous change of flow conditions may appear as a discontinuity in this case.

To complete the discussion of this case it is pointed out that the downstream flow conditions are characterized by inequalities corresponding to (3.43)–(3.47) and (3.49).

5. Shock waves with velocity equilibrium

In this section it is assumed that, by choosing a suitable scale, the relaxation process of the particle velocity has been removed from consideration. Thus both components move with the same velocity and (2.1)–(2.4) reduce to

$$\frac{u}{\nu_M} = \frac{u_0}{\nu_{M0}}, \tag{5.1}$$

$$\eta = \eta_0, \tag{5.2}$$

$$\frac{u^2}{\nu_M}+p = \frac{u_0^2}{\nu_{M0}}+p_0, \tag{5.3}$$

$$h_M+\tfrac{1}{2}u^2 = h_{M0}+\tfrac{1}{2}u_0^2, \tag{5.4}$$

where ν_M is the specific volume of the mixture given by

$$\frac{1}{\nu_M} = \frac{1-\epsilon}{\nu_G}+\epsilon\rho_P, \tag{5.5}$$

and h_M is the enthalpy per unit mass of the mixture

$$h_M = \frac{1}{1+\eta_0}h_G+\frac{\eta_0}{1+\eta_0}\left(c_P T_P+\frac{p}{\rho_P}\right). \tag{5.6}$$

These equations show that the behaviour of the mixture is analogous to that of a single medium in which a thermodynamic relaxation process, such as the relaxation of the temperature of vibration, takes place.

† A derivation will be given in a paper concerning sound waves in a gas–particle flow (Schmitt-von Schubert 1969).

In this case a shock wave appears either as an equilibrium or a partly frozen shock wave, depending on whether the length of the scale is also much greater than the relaxation length of the temperature equilibration. Thus the equations given above are completed by the relation (2.6) or (2.8) for a partly frozen or equilibrium shock wave, respectively.

Taking into account the relation

$$\nu_G = (1+\eta_0)\,\nu_M - \frac{\eta_0}{\rho_P}, \tag{5.7}$$

which results from (5.2) and (5.5) by elimination of ϵ, it is seen that h_M may be written as a function of ν_M and p alone, in both cases. The following definition of an Hugoniot function is suggested by the analogy to the flow of a single medium:

$$H = 2(h_M - h_{M0}) - (p-p_0)\,(\nu_M + \nu_{M0}). \tag{5.8}$$

Hence, the Hugoniot curve $H = 0$ is given in the (ν_M, p)-plane by a single relation.

In this case, unlike that considered in the last section, one may speak of the entropy s_M per unit of mass of the mixture. This quantity is given by the weighted mean of the entropies s_G and s_P,

$$ds_M = \frac{1}{1+\eta_0}\,ds_G + \frac{\eta_0}{1+\eta_0}\,ds_P. \tag{5.9}$$

After some algebra, and using (2.13) and (2.14), it may be concluded from this equation that the following inequalities hold:

$$\left(\frac{\partial p}{\partial \nu_M}\right)_{s_M,\,T_P} < 0, \quad \left(\frac{\partial p}{\partial \nu_M}\right)_{s_M,\,T_P=T_G} < 0, \tag{5.10}$$

$$\left(\frac{\partial s_M}{\partial p}\right)_{\nu_M,\,T_P} > 0, \quad \left(\frac{\partial s_M}{\partial p}\right)_{\nu_M,\,T_P=T_G} > 0. \tag{5.11}$$

Hence, on the assumptions

$$\left(\frac{\partial^2 p}{\partial \nu_M^2}\right)_{s_M,\,T_P} > 0, \quad \left(\frac{\partial^2 p}{\partial \nu_M^2}\right)_{s_M,\,T_P=T_G} > 0, \tag{5.12}$$

the qualitative shape of the isentropes $s_M(\nu_M, p) =$ const. agrees with that which the isentropes $s_G(\nu_G, p) =$ const. of the pure gas have on the assumption

$$\left(\frac{\partial^2 p}{\partial \nu_G^2}\right)_{s_G} > 0. \tag{5.13}$$

The last inequality holds in general for the common gases. Since ν_M is proportional to ν_G, and since in the case of frozen particle temperature the entropy s_M is constant if and only if s_G is constant, the first assumption of (5.12) is identical with (5.13). The second assumption of (5.12), which concerns the case of total equilibrium, is merely analogous to (5.13).

With the use of the quantity s_M the differential of the Hugoniot function takes the form

$$dH = 2T_G\,ds_M + p_0 p^* R\,dQ, \tag{5.14}$$

where

$$R = \nu_{M0}\,\nu_M^*, \tag{5.15}$$

$$dQ = \frac{dp^*}{p^*} - \frac{d\nu_M^*}{\nu_M^*}, \tag{5.16}$$

$$\nu_M^* = \frac{\nu_M}{\nu_{M0}} - 1. \tag{5.17}$$

The curves $Q = \text{const.}$ are straight lines through the origin of the (ν_M^*, p^*)-plane.

Thus the close analogy to a pure gas, studied by Cowan (1958), is demonstrated. Hence one may continue the examination in the same way as Cowan did. The result corresponds to that concerning the pure gas: if and only if the upstream velocity u_0 exceeds the velocity of sound, i.e. if

$$u_0^2 > c_{10}^2 \quad \text{respectively,} \quad u_0^2 > c_0^2, \tag{5.18}$$

where
$$c_1^2 = \frac{(1+E)^2}{1+\eta} a^2, \quad c^2 = \frac{(1+E)^2}{1+\eta} \frac{1+\eta\delta}{1+\gamma\eta\delta} a^2,\dagger \tag{5.19}$$

there exists a uniquely determined partly frozen, respectively equilibrium, compressive shock wave. The upstream values of the flow variables ν_M, p and u are related to their downstream values by inequalities corresponding to (3.44)–(3.46), and thus also corresponding to those which are valid for the pure gas.

Comparison of c_1 and c with b and c_2 in view of $\rho > 2$ leads to the inequalities

$$b^2 > \begin{Bmatrix} c_1^2 \\ c_2^2 \end{Bmatrix} > c^2. \tag{5.20}$$

Hence, an equilibrium shock may exist at the lowest upstream velocity. Partly frozen shocks require a higher value of u_0, but this value still lies below that needed for frozen shocks.

These facts are to be understood in the following way. A change of the flow conditions is possible if the upstream velocity exceeds the equilibrium velocity of sound c_0. This change of state is a continuous one (i.e. a fully dispersed wave) if the upstream velocity is smaller than the frozen velocity of sound b_0, while it consists of a discontinuity followed by a region of continuous change of the flow parameters (i.e. a partly dispersed wave) if the upstream velocity is larger than the frozen velocity of sound. By suitable choice of the scale in each case, the whole change of the flow conditions may appear as an equilibrium shock, or, in those cases where the upstream velocity exceeds one of the partly frozen velocities of sound c_{10} and c_{20}, as a partly frozen shock followed by a region of continuous change of state.

Attention should be paid to the fact that this qualitative behaviour of the mixture is analogous to that of a pure gas in which two thermodynamic relaxation processes are possible. But it must be emphasized that this analogy could not have been anticipated; for, if the constituents of the mixture do not move with the same velocity, there is an essential difference between the equations characterizing the flow of the mixture and those characterizing the flow of such a gas.

† The equation for the partly frozen velocity of sound c_1 was given by Kraiko & Sternin (1965), the equation for the equilibrium velocity of sound c was given by Rudinger (1965).

Appendix A

In order to get a qualitative discussion of the function $\nu_G(\epsilon)$, (3.3) is differentiated:

$$\frac{d\bar{\nu}_G}{d\epsilon} = \frac{1}{(1-\epsilon_0)^2}\left\{\rho_0(\tfrac{1}{2}-\epsilon_0)-(1-\epsilon_0)+\tfrac{3}{2}\rho_0\frac{\epsilon_0^2}{\epsilon^2}-\rho_0\frac{\epsilon_0^2}{\epsilon^3}\right\}, \tag{A 1}$$

$$\frac{d^2\bar{\nu}_G}{d\epsilon^2} = \frac{3\rho_0\epsilon_0^2}{(1-\epsilon_0)^2}\left\{-\frac{1}{\epsilon^3}+\frac{1}{\epsilon^4}\right\}. \tag{A 2}$$

Hence the following table may be realized:

ϵ	$\bar{\nu}_G$	$\dfrac{d\bar{\nu}_G}{d\epsilon}$	$\dfrac{d^2\bar{\nu}_G}{d\epsilon^2}$
$-\infty$	Asymptotically $\sim\beta_0\epsilon$	β_0	-0
± 0	$+\infty$	$\mp\infty$	$+0$
ϵ_0	1	$-\dfrac{\epsilon_0+\rho_0(1-\epsilon_0)}{\epsilon_0(1-\epsilon_0)}$	$\dfrac{3\rho_0}{\epsilon_0^2(1-\epsilon_0)}$
$\frac{1}{2}$	$-\frac{1}{2}\beta_0$	$\beta_0-\dfrac{2\rho_0\epsilon_0^2}{(1-\epsilon_0)^2}$	$\dfrac{24\rho_0\epsilon_0^2}{(1-\epsilon_0)^2}$
1	0	$\frac{1}{2}\alpha_0$	0
>1			<0
$+\infty$	Asymptotically $\sim\beta_0\epsilon$	β_0	$+0$

Here

$$\bar{\nu}_G = \frac{\nu_G}{\nu_{G0}},$$

$$\alpha_0 = \rho_0-\frac{2}{1-\epsilon_0},$$

$$\beta_0 = \frac{1}{(1-\epsilon_0)^2}\{\rho_0(\tfrac{1}{2}-\epsilon_0)-(1-\epsilon_0)\}.$$

The functions $\alpha_0 = 0$ and $\beta_0 = 0$ are shown in figure 9. It may be seen that, on the assumptions (3.11) and (3.12), the following inequalities are valid:

$$\alpha_0 > 0, \quad \beta_0 \geqslant 0.$$

Since $\nu_G(\epsilon)$ does not vanish at more than three real values of ϵ, this function has the qualitative shape shown in figure 10. By consideration of the physical meaning of ϵ and ν_G, it is concluded that only the domain bounded by $\epsilon = 0$ and $\epsilon = \epsilon_z$ with $\epsilon_z \neq 1$ and $\nu_G(\epsilon_z) = 0$ has physical relevance. Thus in this paper 'the function $\nu_G(\epsilon)$' always stands for that part of this function. From this restriction it follows that both $\nu_G(\epsilon)$ and $\epsilon(\nu_G)$ are single-valued functions, and that

$$\frac{d\nu_G}{d\epsilon} < 0, \tag{A 3}$$

$$\frac{d^2\nu_G}{d\epsilon^2} > 0. \tag{A 4}$$

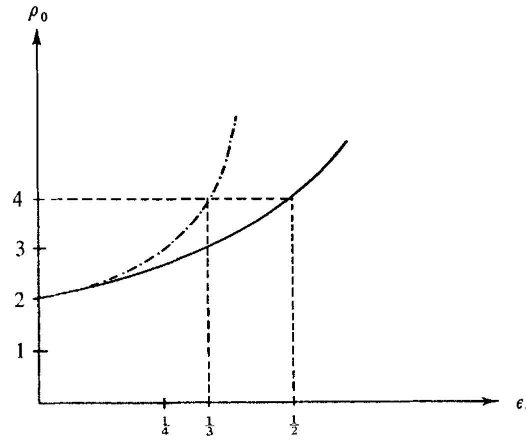

FIGURE 9. The curves $\alpha_0 = 0$ (——) and $\beta_0 = 0$ (-·-).

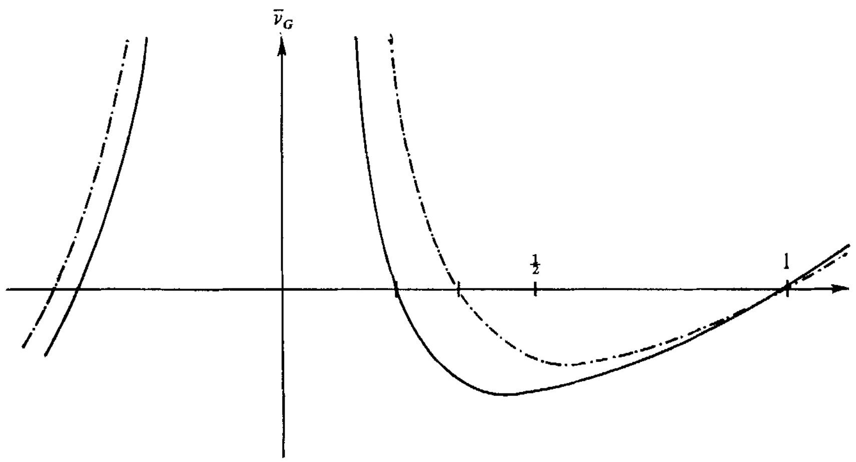

FIGURE 10. Qualitative shape of the function $\bar{\nu}_G(\epsilon)$ for $\beta_0 \lesssim 2\rho_0\{\epsilon_0/(1-\epsilon_0)\}^2$: -·-, ——.

Appendix B

In order to give an estimation of the function $f(\epsilon)$ defined by (3.16), it is first noted that

$$f(0) = \lim_{\epsilon\to 0} \frac{\nu_G(d^2\nu_G/d\epsilon^2)}{(d\nu_G/d\epsilon)^2} = \frac{3}{2}. \tag{B 1}$$

Hence the following definition of a function $g(\epsilon)$ is suggested:

$$\left.\begin{aligned} g(\epsilon) &= \frac{2}{3}\frac{(1-\epsilon_0)^4}{\nu_{G0}^2}\epsilon^6\left(\frac{d\nu_G}{d\epsilon}\right)^2\{f(\epsilon)-\tfrac{3}{2}\} \\ &= \frac{2}{3}\frac{(1-\epsilon_0)^4}{\nu_{G0}^2}\epsilon^6\left\{\nu_G\frac{d^2\nu_G}{d\epsilon^2}-\frac{3}{2}\left(\frac{d\nu_G}{d\epsilon}\right)^2\right\}. \end{aligned}\right\} \tag{B 2}$$

This function may be rearranged in the following way:

$$g(\epsilon) = \epsilon\{g_1(\epsilon)+g_2(\epsilon)\}, \tag{B 3}$$

where
$$g_1(\epsilon) = \alpha_1\epsilon^3+\alpha_2\epsilon^5,$$
$$g_2(\epsilon) = -\alpha_3(1-3\epsilon)\,\epsilon-\rho_0\epsilon_0^4(1-\tfrac{11}{4}\epsilon)-2\rho_0^2\epsilon_0^4,$$
$$\alpha_1 = -5\rho_0\epsilon_0^2\{\rho_0(\tfrac{1}{2}-\epsilon_0)-(1-\epsilon_0)\},$$
$$\alpha_2 = -\{\rho_0(\tfrac{1}{2}-\epsilon_0)-(1-\epsilon_0)\}^2,$$
$$\alpha_3 = 2\rho_0\epsilon_0^2\{\rho_0(\tfrac{1}{2}-\epsilon_0)-(1-\epsilon_0)\}.$$

The assumptions (3.11) and (3.12) lead to
$$\alpha_1 < 0, \quad \alpha_3 > 0.$$

Using these inequalities and the assumption (3.17) one may verify that
$$g_1(\epsilon) < 0, \quad g_2(\epsilon) < 0;$$
hence
$$g(\epsilon) < 0. \tag{B 4}$$

The relations (B 2) and (B 4) lead to the conclusion that
$$f(\epsilon) < \tfrac{3}{2}. \tag{B 5}$$

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